

Monday, October 13, Lecture 19: Cauchy Consequences, Liouville's Thm

Last time...

Thm (Cauchy's thm)

Let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic on a simply connected region Ω , and let $\gamma: [a, b] \rightarrow \Omega$ be a cts, piecewise smooth loop. Then

$$\int_{\gamma} f(z) dz = 0.$$

Cor: if γ and $\tilde{\gamma}$ have the same orientation and endpoints, then



$$\int_{\gamma} f(z) dz = \int_{\tilde{\gamma}} f(z) dz$$

The simply connected assumption on Ω is crucial!

Ex: (extremely important).

Let $f(z) = \frac{1}{z}$, and $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Then

$$\begin{aligned} \int_{\gamma} \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{(e^{it})} \cdot i e^{it} dt \\ &= \int_0^{2\pi} i dt \\ &= 2\pi i. \end{aligned}$$

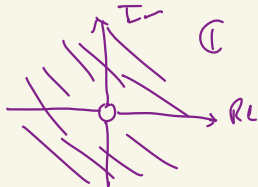


Why doesn't Cauchy's thm apply?

$\frac{1}{z}$ has a singularity at $z=0$, so the domain Ω

has a hole and is not simply connected:

$$\Omega = \mathbb{C} \setminus \{0\},$$



HW: Primer to residue thm.

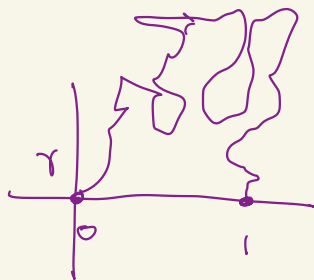
This is a surprisingly powerful theorem.

We may use it to "deform" line integrals.

Ex: Deformation

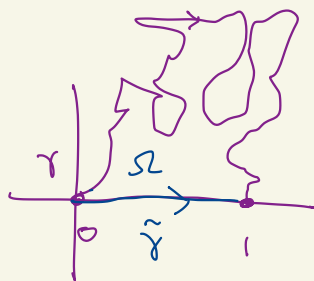
Compute

$$\int_{\gamma} e^z dz \text{ along}$$



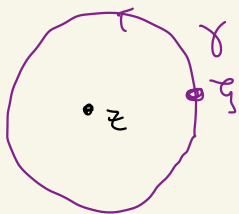
By Cauchy's thm, since e^z holomorphic inside Ω ,

$$\begin{aligned} \int_{\gamma} e^z dz &= \int_{\tilde{\gamma}} e^z dz \\ &= \int_0^1 e^x dx \\ &= e^1 - 1. \end{aligned}$$



Cauchy's integral formula

How to handle holes? well...



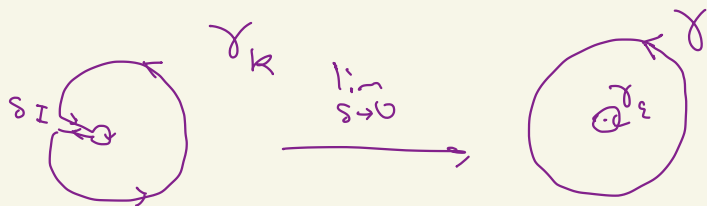
We can't immediately

$$\text{calculate } \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

since $\Omega = \mathbb{C} \setminus \{z\}$ is not simply connected.

← singularity at $\zeta = z$.

But...



Cauchy

$$0 = \int_{\gamma_R} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta + \underbrace{\int_{\gamma_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta}$$

Since f holomorphic, f is continuous, so as $z \rightarrow 0$, $f(\zeta) \approx f(z)$ for all ζ on γ_ϵ

$$\approx \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta + f(z) \underbrace{\int_{\gamma_\epsilon} \frac{1}{\zeta - z} d\zeta}$$

This is just like $\int \frac{1}{\zeta}$, but offset and backwards.



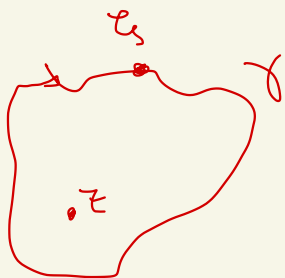
$$= \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta + f(z) (-2\pi i)$$

Thm: Cauchy's Integral Formula

Let f be holomorphic on simply connected Ω and γ a loop in Ω .

Let z be inside γ . Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi$$



$f(z)$ is completely determined by its boundary values!

How crazy is that??

• Notice: if z is outside γ , then $\frac{f(\xi)}{\xi - z}$

is holomorphic inside γ ,
so

$$0 = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi$$

By Cauchy's theorem.

• Warning! Thm does not apply if f is not holo,
eg. $|z|$ or $\bar{z}$.

We can do even more.

Notice: $\frac{d}{dz} \frac{1}{(y-z)} = \frac{1}{(y-z)^2}.$

$$\frac{d^n}{dz^n} f(z) = \frac{1}{2\pi i} \int_{\gamma} f(y) \frac{d^n}{dz^n} \left(\frac{1}{y-z} \right) dy$$

(*) Doing this
requires some
careful math.

$$= \frac{1}{2\pi i} \int_{\gamma} f(y) n! \frac{1}{(y-z)^{n+1}} dy$$

Then (Cauchy's Integral Formula for Derivatives)

Let Ω be simply connected and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Then for a point z inside a loop γ ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(y)}{(y-z)^{n+1}} dy.$$

n^{th} derivative, so $f^{(0)}(z) = f(z)$
 $f^{(1)}(z) = f'(z)$

Ex: Compute, with γ = unit circle.

$$\int_{\gamma} \frac{e^z}{z^2} dz.$$

Soln: Observe that e^z is holomorphic, so this is of the form

$$\int_{\gamma} \frac{f(z)}{(z-0)^2} dz = \frac{2\pi i}{1!} f'(0) = \frac{2\pi i}{1} e^0 = 2\pi i.$$

Prop: Cauchy Inequalities

Let f holomorphic on Ω and let $\overline{D_R(z_0)} \subseteq \Omega$ with boundary γ .

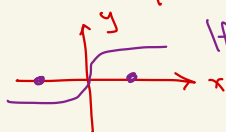
Sps $|f(\zeta)| \leq M$ for all $\zeta \in \gamma$. Then for all $k=1,2,\dots$



$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M.$$

• So: derivatives of holomorphic functions cannot be "too big".

• Very different from real analysis! e.g. $f(x) = x^{1/3}$ on $[-1,1]$



$|f(x)| \leq 1$ on $\gamma = [-1,1]$,

but $f'(x)$ is unbounded
 $f'(x) = \frac{1}{3}x^{-2/3}$.

pf:

$$|f^{(k)}(z_0)| = \left| \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right|$$

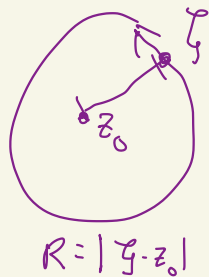
$$\leq \frac{k!}{2\pi} \int_{\gamma} \frac{|f(\zeta)|}{|\zeta - z_0|^{k+1}} d\zeta$$

$|f(\zeta)| \leq M$

$$\leq \frac{k!}{2\pi} \int_{\gamma} \frac{M}{R^{k+1}} d\zeta$$

$$= \frac{k! M}{2\pi R^{k+1}} \underbrace{(2\pi R)}_{= \text{length}(\gamma)}$$

$$= \frac{k!}{R^k} M.$$



Def: Entire

We call a function $f: \mathbb{C} \rightarrow \mathbb{C}$ entire if f is holomorphic on all of $\mathbb{C}$.

Ex: polynomials, exponential series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Nonex: $\frac{1}{z}$, geometric series $\sum_{n=0}^{\infty} z^n$

Thm (Liouville's Thm)

If f is entire and bounded, i.e. $\exists M$ with $|f(z)| \leq M \forall z \in \mathbb{C}$, then f is constant.

Pf:

By Cauchy Inequalities, for any $z \in \mathbb{C}$, $|f'(z)| \leq \frac{M}{R}$.

Now, take $\lim_{R \rightarrow \infty} \frac{M}{R} = 0$, so $f'(z) = 0$, $\square$

This one is dramatic. There are many bounded nonconstant functions on $\mathbb{R}$:

• $\sin x, \cos x$



• Gaussian e^{-x^2}



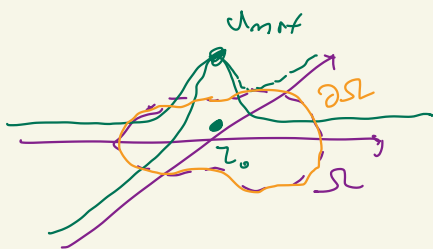
Bounded in all directions is a strong condition!

• $\sin(ix) = \frac{e^{-x} - e^x}{2i}$ is unbounded

• $e^{-(ix)^2} = e^{x^2}$ is unbounded

One more big theorem, closely related to optimization.

• For real functions $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ on a bounded domain Ω , easy to have "interior maxima":



z_0 is inside Ω ,
not on boundary $\partial\Omega$

Surprising fact: this can never happen if u is a nonconstant harmonic function, i.e.